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Properties of a Transient Queue

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1. INTRODUCTION AND SUMMARY

We consider in this paper a queueing process similar to one studied by Borel [1] in 1942 and, in greater generality, by B. McMillan and J. Riordan [4] in 1957. Let $E_1, E_2, \ldots$ denote, respectively, members of an infinite queue, where the subscript indicates the order in line. We think of these elements as being located, initially, at points of the set

$$R^+ = \left\{ x : 0 \leq x < \infty \right\},$$

the distances between adjacent elements being stochastically determined. At time $t = 0$ the entire queue begins to move toward the origin, 0, the unit of distance being chosen so that the motion takes place with unit velocity. Simultaneously, a service facility undertakes to service the lead element, E_1. The service discipline has the following properties:

1) The time required for completion of service is a random variable whose distribution is independent of the element in service.
2) If service on an element E_j is completed before (or simultaneous with) the arrival of E_j at 0, the server retracts instantaneously to E_{j+1}.
3) If an element E_j arrives at 0 before service on it has been completed, at the time of arrival the server suffers a probability α, $0 \leq \alpha \leq 1$, of being instantaneously disabled (absorbed). If he is disabled, the process terminates; if not, the server abandons the attempt to complete service on E_j and retracts without loss of time to E_{j+1}.

To describe the process more precisely, let b denote the location of E_1 at time $t = 0$. The distance between E_j and E_{j+1} is a finite-valued random variable η_j with distribution function

$$P\left\{\eta_j \leq x\right\} = G(x) \quad , \quad j = 1,2,\ldots \quad .$$

No assumption is made about the form of G, but it is clear from its definition that

(i) $G(x) = 0 \quad , \quad x < 0,$

(ii) $G(x+0) = G(x),$

and

(iii) $G(\infty) = \lim_{x\to\infty} G(x) = 1.$

The time required for completion of service on E_j is a random variable ξ_j which may, with positive probability, be infinite. Its distribution function

$$P\left\{\xi_j \leq x\right\} = F(x) \quad , \quad j = 1,2,\ldots,$$

is arbitrary. Evidently F, like G, has properties (i) and (ii); but (iii) is replaced by the weaker constraint

(iii)' $\qquad F(\infty) = \lim_{x\to\infty} F(x) \leq 1.$

The random variables $\left\{\xi_j,\eta_j\right\}_{j=1}^{\infty}$ are all assumed to be mutually independent.

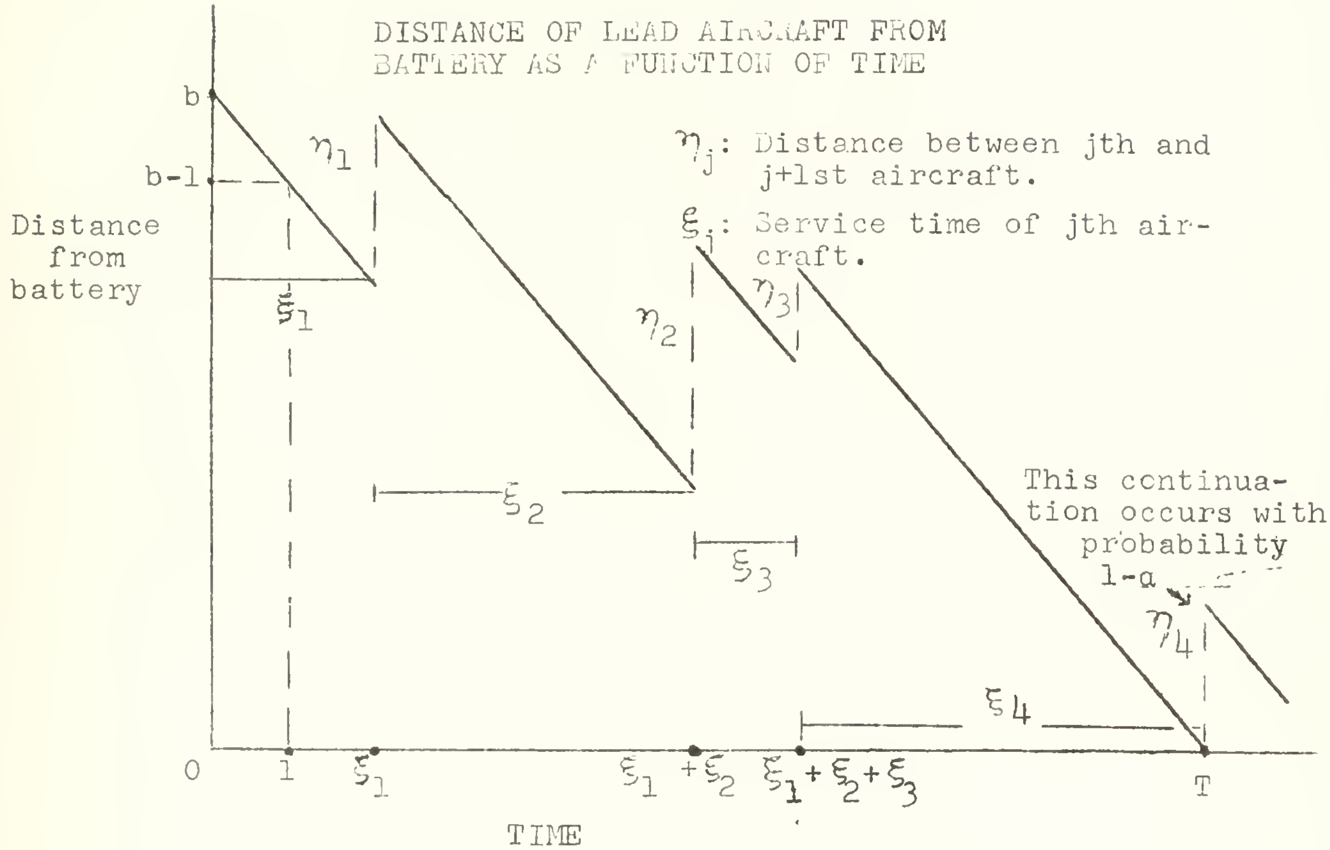
The process we have described may be regarded as a crude model of the operation of a missile battery against a file of aircraft, the battery being the server and the aircraft his

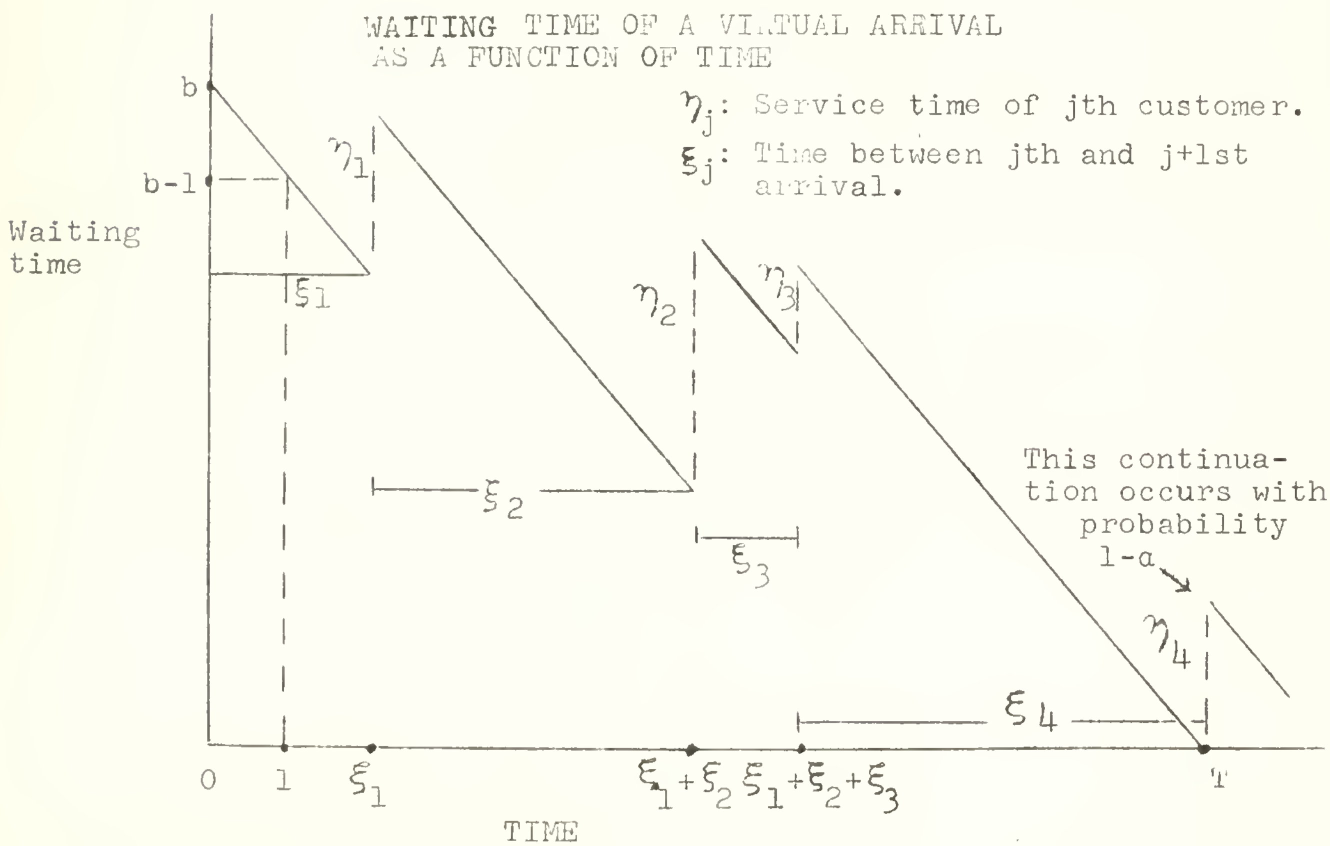
(ungrateful) customers. Serving an aircraft means destroying it, and arrival of an aircraft at O means that it is in position to counter-attack the battery. The assumptions made above imply that the counter-attack is instantaneous and for all aircraft has a fixed probability α of succeeding. The initial distance b can be regarded as the point at which a warning system alerts the battery to the incoming attack. The assumption that an aircraft arriving at O is abandoned by the server in favor of an incoming aircraft can be interpreted to mean that the battery doesn't "turn around". Finally, the provision that the service-time distribution F may have a jump at infinity is an allowance for the possible imperfection in our weapons systems.

In the problem described by B. McMillan and J. Riordan O is a true absorbing barrier, which corresponds in the present context to the case $\alpha = 1$. In that case, as they observed, the problem of finding the probability distribution of the number of elements served before absorption is equivalent to finding the number of elements served in the busy period of an associated, conventional single server system. When α is arbitrary, a similar identification is possible, but the associated busy-period problem seems somewhat contrived if $\alpha < 1$. Thus, consider an ordinary single-server queue in which the service time of the first customer is the constant b, and the service times of succeeding customers are independent, identically distributed random variables having the distribution function G. Let the origin of the time axis be the arrival time of the first

customer, and suppose that the inter-arrival times are independent, identically distributed random variables having the distribution function F. Suppose that at the end of each busy period there is a probability α that the server becomes ill or disenchanted with his duties and serves no further customers. With probability $1-\alpha$ he continues operations, in which case a customer arrives instantaneously. (There can be no slack periods.) The duality between this problem and the moving queue is exhibited in figures 1 and 2, which are identical in form but have different labellings. These figures summarize, respectively, activities during one busy period of the single-server queue and between successive arrivals at 0 in the moving queue. We observe that if v is the number of customers served during a busy period, and ρ is the number of elements served between two successive arrivals at 0, then

$$v = \rho + 1.$$


DISTANCE OF LEAD AIRCRAFT FROM BATTERY AS A FUNCTION OF TIME
η_j: Distance between jth and j+1st aircraft.
ξ_j: Service time of jth aircraft.
Distance from battery
b
b-1
η_1
η_2
η_3
η_4
ξ_1
ξ_2
ξ_3
ξ_4
This continuation occurs with probability 1-α
0
1
ξ_1
$\xi_1+\xi_2$
$\xi_1+\xi_2+\xi_3$
T
TIME


WAITING TIME OF A VIRTUAL ARRIVAL AS A FUNCTION OF TIME
η_j: Service time of jth customer.
ξ_j: Time between jth and j+1st arrival.
Waiting time
b
b-1
η_1
η_2
η_3
η_4
ξ_1
ξ_2
ξ_3
ξ_4
This continuation occurs with probability 1-α
0
1
ξ_1
$\xi_1+\xi_2$
$\xi_1+\xi_2+\xi_3$
T
TIME

Let p denote the probability of serving infinitely many elements in the queue. If $\alpha > 0$, it is easy to see that p is, in fact, the probability of serving all but a finite number of elements. In turn this can be identified with the probability that the server survives forever. We regard p as one measure of the effectiveness of the service facility. It is clear that p depends on the initial distance b, the absorption probability α, and the distribution functions F, G. We shall study the dependence of p on these quantities. Throughout most of the paper α, F, and G are fixed (though arbitrary), and we adopt a notation which emphasizes the role of b, namely, the value of p at (b,α,F,G) is denoted by $p(b)$. In the few sections where it is necessary to exhibit explicitly the dependence on α, we write $p^{(\alpha)}(b)$ instead of $p(b)$.

In the aircraft vs. missile battery model it is intuitive that the more accurate the weapons of the attacker the less effective the defense; in other words, that p decreases as α increases. Another natural conjecture, suggested by the identification of b with the range of a warning system, is that p increases as b increases. Finally, in the special case when the aircraft are equally spaced d units apart, we would expect that p is an increasing function of d. These conjectures are made more precise and proved in section 3.

The main results, stated in section 4, concern the dependence of p on F and G. Since we permit these functions to be arbitrary, we do not attempt to exhibit p explicitly in terms of them. Instead we determine a criterion for classifying pairs of distributions (F,G) into two categories: those

for which $p = 0$ for all values of (b,α) and those for which $p > 0$ for some choice of (b,α). As will be seen, this criterion does not involve the details of the distributions but only the relation between their first moments. In the case when $p > 0$ for some value of (b,α), we study the behavior of p for large values of b.

2. FUNCTIONAL EQUATIONS

The process we have described has a certain recursive character which leads to a functional equation for p: Once an element has been served or has arrived at 0, we may think of the process as terminated and of a new one originating, although from a possibly different initial distance. Let

A_1 = "E_1 and infinitely many E_n, $n > 1$, are served,"

and

A_2 = "E_1 is not served, E_1 fails to disable the server, and infinitely many E_n are served, $n > 1$."

Then, if E_1 is initially at b,

$$P\{A_1 | \xi_1 = t, \eta_1 = x\} = p(b-t+x),$$
$$P\{A_2 | \eta_1 = x\} = (1-F(b))(1-\alpha)p(x),$$

and

$$p(b) = P(A_1) + P(A_2).$$

Letting I_x denote the closed interval $[0,x]$, it follows that

$$(1) \quad p(b) = \int_{R^+}\int_{I_b} p(b-t+x)dF(t)dG(x)+(1-F(b))(1-\alpha)\int_{R^+}p(x)dG(x).$$

This equation, being homogeneous in p, has the solution $p \equiv 0$ for any pair of distributions F,G; hence it does not in general uniquely characterize p. However, in the sequel we shall investigate conditions under which $p \equiv 0$ is the unique solution.

It is easy to see that equation (1) cannot alone be used to establish the monotonicity properties of p conjectured in § 1; for, it has non-monotonic solutions, even when supplemented by the constraint $0 \leq p \leq 1$. As an example, let $F(0) = 1$ and $G(d) - G(d-0) = 1$, $d > 0$. Then (1) reduces to

$$p(b) = p(b+d),$$

which is satisfied by any function defined arbitrarily in the interval $[0,d]$ and continued periodically with period d.

To facilitate the study of p we consider the related quantities $\{p_k\}_{k=0}^{\infty}$, where p_k is the probability of serving at least k elements. These are functions of the same arguments as p, and we adopt for them the same notational conventions. We recover p from the sequence $\{p_k\}_{k=0}^{\infty}$ by means of the limit relation

(2) $$p = \lim_{k\to\infty} p_k.$$

A functional equation for p_k is obtained by the same type of argument that led to (1). Thus,

$$p_k(b) = \int_{R^+}\int_{I_b} p_{k-1}(b-t+x)dF(t)dG(x)$$

(3) $$+ (1-F(b))(1-\alpha)\int_{R^+} p_k(x)dG(x), \; k \geq 1,$$

$$p_0 \equiv 1.$$

Integrating both sides of (3) with respect to G and solving for the integral, $\int_{R^+} p_k(x)dG(x)$, we find

(4) $$\int_{R^+} p_k(b)dG(b) = \frac{\int_{R^+}\int_{R^+}\int_{I_b} p_{k-1}(b-t+x)dF(t)dG(x)dG(b)}{1-(1-\alpha)(1-\int_{R^+}F(b)dG(b))},$$

provided that

(5) $$\alpha + \int_{R^+} F(b)dG(b) > 0.$$

It follows that, if (5) holds, (3) has a unique solution.

We shall show that even if (5) is violated, (3) has a unique solution: Assume that

$$\alpha + \int_{R^+} F(b)dG(b) = 0,$$

or even less restrictively, that

$$\int_{R^+} F(b)dG(b) = 0.$$

If F does not have a unit jump at infinity, this equation implies that there is a number b* with the properties:

(i) $$F(b^*-0) = 0, \; G(b^*) = 1$$

and

(ii) $$F(b^*)[G(b^*)-G(b^*-0)] = 0.$$

Hence, we have

(i)' $$P\{\xi_i \geq b^*, \; i = 1,2,\ldots\} = P\{\eta_i \leq b^*, \; i = 1,2,\ldots\} = 1$$

and

(ii)' $$P\{\xi_i = b^* \text{ and } \eta_j = b^* \text{ for some } i,j\} = 0.$$

From the definitions of ξ_i and η_i it follows that

$$p_k(x) = 0 \quad , \quad k \geq 1, \; x \leq b^*,$$

and accordingly,

$$\int_{R^+} p_k(x)dG(x) = \int_{I_{b^*}} p_k(x)dG(x) + \int_{R^+ - I_b{}^*} p_k(x)dG(x) = 0.$$

Substituting into (3) we obtain the recursion relation

$$p_k(b) = \int_{R^+} \int_{I_b} p_{k-1}(b-t+x)dF(t)dG(x) \quad , \quad k \geq 1,$$

(6)

$$p_0 \equiv 1,$$

which has a unique solution.

In the exceptional case when F induces the measure with unit mass at infinity, i.e.,

$$P\left\{\xi_i = \infty,\ i = 1,2,\ldots\right\} = 1,$$

it is clear from the interpretation of ξ_i that

$$p_0 \equiv 1$$

and

$$p_k \equiv 0 \quad , \quad k \geq 1.$$

Thus in all cases the probabilities $\left\{p_k\right\}_{k=0}^{\infty}$ are uniquely determined by (3).

3. MONOTONICITY THEOREMS

THEOREM 1. The probabilities $\left\{p_k\right\}_{k=0}^{\infty}$ and p are non-increasing functions of α in the interval $0 \leq \alpha \leq 1$.

Proof: The theorem follows immediately from (3) and (4) by induction on k and use of the limit relation (2).

To simplify notations and arguments in the rest of this section we introduce the transform

$$(7)\qquad P_k(y) = \begin{cases} \int_{R^+} p_k(y+x)\,dG(x) & , \quad y \geq 0 \\ p_k(0) & , \quad y < 0. \end{cases}$$

Clearly,

$$|P_k(y)| \leq 1,\ P_0(y) \equiv 1, \text{ and } P_{k+1} \leq P_k.$$

In terms of P_k the functional equation (3) for p_k takes the form

$$(8)\qquad \begin{aligned} p_k(b) &= \int_{I_b} P_{k-1}(b-t)\,dF(t) + (1-\alpha)(1-F(b))P_k(0),\ k \geq 1 \\ p_0(b) &\equiv P_0(b) \equiv 1. \end{aligned}$$

LEMMA. The functions $p_k(y)$ and $P_k(y)$ are right-continuous at each point y in their respective domains of definition.

Proof: Put

$$J_k(y) = \int_{I_y} P_k(y-t)\,dF(t).$$

Then, setting $I_{y,y'} = I_{y'} - I_y = (y,y']$, $y' > y$, we have

$$(9)\qquad \begin{aligned} J_k(y')-J_k(y) &= \int_{I_y} P_k(y'-t)-P_k(y-t)\,dF(t) \\ &+ \int_{I_{y,y'}} P_k(y'-t)\,dF(t). \end{aligned}$$

Assume that for an arbitrary but fixed value k, $p_k(y)$ is right-continuous at each $y \varepsilon R^+$. It follows from (7) and the Lebesgue bounded convergence theorem that $P_k(y)$ is also right-continuous. Since

$$|J_k(y')-J_k(y)| \leq \int_{I_y} |P_k(y'-t)-P_k(y-t)|dF(t)+F(y')-F(y),$$

again using bounded convergence we see that $J_k(y)$ is right-continuous. But from (8),

$$p_{k+1}(y) = J_k(y) + (1-\alpha)(1-F(y))P_{k+1}(0).$$

Hence $p_{k+1}(y)$ is right-continuous, which, as we have already observed, implies that $P_{k+1}(y)$ has the same property. Observing that $p_0(y)$ is (trivially) right-continuous, it follows by induction that the theorem is true for $y \varepsilon R^+$. Since for $y < 0$,

$$P_k(y) = P_k(0),$$

the proof is complete.

With the aid of this lemma we establish next the monotonic character of the dependence of $p_k(b)$ and $p(b)$ on b.

THEOREM 2. The probabilities $p_k(b)$ and $p(b)$ are non-decreasing functions of b.

Proof: We observe from (7) that the monotonicity of p_k entails that of P_k. Making the inductive hypothesis that $p_{k-1}(b)$ is non-decreasing in b, it is not immediately clear from (8) whether $p_k(b)$ has the same property; for the first term on

the right-hand side of that equation is non-decreasing, while the second is non-increasing. To compare them we integrate the first term by parts,[1] obtaining after an obvious change of variables,

$$(10)\quad \int_{I_b} P_{k-1}(b-t)dF(t) = \int_{I_b} F(b-t)dP_{k-1}(t) + F(b)P_{k-1}(0).$$

Substituting this result into (8) we find that

$$p_k(b) = \int_{I_b} F(b-t)dP_{k-1}(t)$$

$$+ F(b)[P_{k-1}(0)-(1-\alpha)P_k(0)]+(1-\alpha)P_k(0).$$

---

(1) Here we use a version of the theorem on integration by parts due to L. C. Young [5] whose proof shows that if Φ and ψ are of bounded variation in $[a,b]$, and one of them is left-continuous while the other is right-continuous, then

$$(*)\qquad \int_{[a,b]} \Phi d\psi + \int_{[a,b]} \psi d\Phi = \Phi(b+)\psi(b+)-\Phi(a-)\psi(a-).$$

In our application $P_{k-1}(b-t)$ is, by inductive hypothesis, monotonic in t and, by the previous lemma, left-continuous in t. F(t) is, of course, right-continuous by its definition.

We remark that (*) may fail to hold if Φ and ψ are continuous from the same side. Moreover, it is meaningless unless Φ and ψ are defined in an open interval containing $[a,b]$. It was with this application in mind that we defined $P_k(y)$ for all real y. It is interesting to note that the definition

$$P_k(y) = P_k(0) \quad , \quad y < 0,$$

seems essential to the proof of theorem 2.

From the monotonicity of P_{k-1} it follows that the integral is a non-decreasing function of b. Since $P_{k-1} \geq P_k$, the same is true of the second term. Hence $p_k(b)$ is non-decreasing in b. Since $p_0 \equiv 1$, the induction is complete.

The monotonicity of p(b) follows from the relation

$$p(b) = \lim_{k\to\infty} p_k(b),$$

which completes the proof.

Suppose next that the spacing distribution G degenerates at a point $d \in R^+$, i.e.,

$$(11) \qquad G(d) - G(d-0) = 1.$$

To emphasize the dependence of p and p_k on d we write $p_k(b,d)$ and $p(b,d)$, respectively, in place of $p_k(b)$ and $p(b)$.

THEOREM 3. The probabilities $p_k(b,d)$ and $p(b,d)$ are non-decreasing functions of d.

Proof: If G has the form given in (11), and if

$$\int_{R^+} F(b)dG(b) = F(d) > 0,$$

equations (3) and (4) reduce to

$$(12) \quad p_k(b,d) = \int_{I_b} p_{k-1}(b+d-t,d)dF(t)+(1-\alpha)(1-F(b))p_k(d,d),$$

where

$$p_k(d,d) = \frac{\int_{I_d} p_{k-1}(2d-t,d)dF(t)}{1-(1-\alpha)(1-F(d))}.$$

In the exceptional case when

$$\int_{R^+} F(b)dG(b) = F(d) = 0,$$

as we have seen in (6), only the first term on the right-hand side of (12) is present.

As in the preceding theorems, the proof is inductive, starting with the observation that $p_0(b,d)$ is non-decreasing in its second argument. Assume this is true of p_{k-1}. Since, by theorem 2, p_{k-1} is a non-decreasing function of its first argument, the integral in (12) is non-decreasing in d.

It remains to show that $p_k(d,d)$ is non-decreasing in d. Choose any number $d' > d$. Then

$$p_k(d'd') = \frac{\int_{I_{d'}} p_{k-1}(2d'-t,d')dF(t)}{1-(1-\alpha)(1-F(d'))}$$

$$= \frac{\int_{I_d} p_{k-1}(2d'-t,d')dF(t)+\int_{I_{d,d'}} p_{k-1}(2d'-t,d')dF(t)}{1-(1-\alpha)(1-F(d'))}$$

$$\geq \frac{\int_{I_d} p_{k-1}(2d-t,d)dF(t)+p_{k-1}(d',d)(F(d')-F(d))}{1-(1-\alpha)(1-F(d'))}$$

Hence,

$$p_k(d',d')-p_k(d,d) \geq \frac{p_{k-1}(d',d)(F(d')-F(d))}{1-(1-\alpha)(1-F(d'))}$$

$$+\int_{I_d} p_{k-1}(2d-t,d)dF(t)\left[\frac{1}{1-(1-\alpha)(1-F(d'))}-\frac{1}{1-(1-\alpha)(1-F(d))}\right]$$

$$= \frac{F(d')-F(d)}{1-(1-\alpha)(1-F(d'))}\left[p_{k-1}(d',d) - \frac{(1-\alpha)\int_{I_d} p_{k-1}(2d-t,d)dF(t)}{1-(1-\alpha)(1-F(d))}\right] =$$

$$\frac{F(d')-F(d)}{1-(1-\alpha)(1-F(d'))}\left[p_{k-1}(d',d) - (1-\alpha)p_k(d,d)\right].$$

Since

$$p_{k-1}(d',d) \geq p_k(d',d) \geq p_k(d,d),$$

it follows that

$$p_k(d',d') \geq p_k(d,d),$$

which completes the proof for the probabilities $\{p_k\}$. The theorem then follows for p from the limit relation (2).

4. DEPENDENCE OF p ON THE SERVICE TIME AND SPACING DISTRIBUTIONS

In the next group of theorems we determine the pairs of distributions (F,G) for which $p \equiv 0$ and those for which $p > 0$ for some choice of (b,α). First we consider a "degenerate" case (theorem 4), only of peripheral interest, when $F(\infty) < 1$ and $\alpha > 0$. The study of the behavior of p when $\alpha > 0$ and $F(\infty) = 1$ is described fully in theorems 5-9. In theorems 10 and 11 we complete the study by considering the "boundary" case $\alpha = 0$. We recall that it is assumed throughout that $G(\infty) = 1$.

For the tails of the distributions F and G we use, respectively, the notations

$$R(x) = 1-F(x)$$

and

$$T(x) = 1-G(x).$$

We denote the expectations by

$$\bar{\xi} = \begin{cases} \int_{R^+} x\,dF(x) = \int_{R^+} R(x)\,dx & \text{if} \quad F(\infty) = 1 \\ \infty & \text{otherwise} \end{cases}$$

and

$$\bar{\eta} = \int_{R^+} x\,dG(x) = \int_{R^+} T(x)\,dx.$$

THEOREM 4. If $\alpha > 0$ and $F(\infty) < 1$, then

$$p(b) = 0$$

for all $b \geq 0$.

Proof: Put

$$\mu(b) = p(b) - (1-\alpha) \int_{R^+} p(x)\,dG(x).$$

If $p(b) = 0$ for all b, it is clear that $\mu(b) = 0$ for all b. Conversely, suppose that $\mu(b) = 0$ for all b. Then,

$$p(b) = (1-\alpha) \int_{R^+} p(x)\,dG(x).$$

Hence p is independent of b. Setting $p(b) = C_\alpha$ we have

$$C_\alpha = (1-\alpha) \int_{R^+} C_\alpha\,dG(x) = (1-\alpha) C_\alpha.$$

Thus $C_\alpha = 0$, and the relations $p(b) = 0$ for all b and $\mu(h) = 0$ for all b are equivalent.

Substituting μ for p in (1) we obtain the equation

$$(13) \qquad \mu(b) = \int_{R^+} \int_{I_b} \mu(b-t+x)\,dF(t)\,dG(x).$$

Since

$$|\mu(b)| = |p(b)-(1-\alpha)\int_{R^+} p(x)dG(x)| \leq 1,$$

we have from (13)

$$|\mu(b)| \leq F(b) \leq F(\infty).$$

Inserting this bound into (13), we obtain the inequality

$$|\mu(b)| \leq F^2(\infty).$$

After n iterations we find that

$$|\mu(b)| \leq F^n(\infty).$$

Since $F(\infty) < 1$, and n is arbitrary, the theorem is proved.

THEOREM 5. If $\alpha > 0$, $\bar{\eta} < \infty$, and $\bar{\eta} \leq \bar{\xi}$, then $p(b) = 0$ identically in b unless there is a real number β such that

$$F(\beta)-F(\beta-0) = G(\beta)-G(\beta-0) = 1.$$

In the latter case, $p(b) = 1$ for all $b \geq \beta$.

Proof: If $F(\infty) < 1$, the result follows from the preceding theorem. Therefore we assume throughout the proof that $F(\infty) = 1$.

To begin with we note that the function $\mu(b)$ defined in the proof of theorem 4 inherits from $p(b)$ the following properties:

(i) $\mu(b) \leq \mu(b^*)$ if $b \leq b^*$,

(ii) $\mu(b+0) = \mu(b)$

(iii) $|\mu(b)| \leq 1$.

In fact (iii) can be sharpened by observing that

$$\mu(b) = \int_{R^+}\int_{I_b} p(b-t+x)dF(t)dG(x) - (1-\alpha)F(b)\int_{R^+} p(x)dG(x)$$

$$\geq \alpha p(0)F(b) \geq 0.$$

Hence,

(iii)' $\quad 0 \leq \mu(b) \leq 1.$

For $s > 0$ we define

$$\tilde{F}(x) = \int_{R^+} e^{-st} dF(t),$$

$$(14) \qquad \phi_s(x) = \begin{cases} \int_{R^+} e^{-st}\mu(t+x)dt & \text{if} \quad x \geq 0 \\ \phi_s(0) & \text{otherwise,} \end{cases}$$

and

$$\tilde{\mu}(s) = \int_{R^+} e^{-st}\mu(t)dt = \phi_s(0).$$

Since $\mu(t+x)$ is non-decreasing in x, the same is true of $\phi_s(x)$. Moreover, it follows from the Lebesgue bounded convergence theorem that $\phi_s(x)$ is right-continuous at each point x, and

$$\phi_s(\infty) = \lim_{x\to\infty} \phi_s(x) = \frac{1}{s}\lim_{x\to\infty} \mu(x) = \frac{1}{s}\mu(\infty).$$

From (13) and Fubini's theorem we see that these transforms satisfy the equation

$$(15) \qquad \tilde{\mu}(s) = \int_{R^+}\int_{R^+}\int_{I_b} \mu(b-t+x)e^{-sb}dF(t)dG(x)db$$

$$= \int_{R^+} e^{-st}dF(t)\int_{R^+}\left(\int_{R^+}\mu(b+x)e^{-sb}db\right)dG(x)$$

$$= \tilde{F}(s)\int_{R^+}\phi_s(x)dG(x).$$

Since both Φ_s and G are right-continuous, integration by parts of $\int_{R^+}\Phi_s(x)dG(x)$ is not justified. However, by "redefining" G to be left-continuous, i.e., setting

$$G^*(x) = \begin{cases} G(x) & \text{at continuity points } x \text{ of } G \\ G(x-0) & \text{otherwise} \end{cases}$$

we obtain the equation

$$(16) \qquad \tilde{\mu}(s) = \tilde{F}(s)\int_{R^+}\Phi_s(x)dG(x) = \tilde{F}(s)\int_{R^+}\Phi_s(x)dG^*(x),$$

in which integration by parts is valid. Thus,

$$(17) \qquad \tilde{\mu}(s) = \tilde{F}(s)\left(\Phi_s(\infty) - \int_{R^+}G^*(x)d\Phi_s(x)\right).$$

Setting

$$T^*(x) = 1 - G^*(x),$$

it follows that

$$(18) \qquad \tilde{\mu}(s) = \tilde{F}(s)\left(\Phi_s(0) + \int_{R^+}T^*(x)d\Phi_s(x)\right)$$

$$= \tilde{F}(s)\left(\tilde{\mu}(s) + \int_{R^+}T^*(x)d\Phi_s(x)\right).$$

Now let

$$\tilde{R}(s) = \int_{R^+}R(t)e^{-st}dt \quad , \quad s > 0.$$

Then

$$\tilde{F}(s) = \int_{R^+}e^{-st}dF(t) = s\int_{R^+}F(t)e^{-st}dt = 1 - s\tilde{R}(s)$$

Hence from (18) we obtain the equation

$$(19) \qquad s\tilde{\mu}(s)\tilde{R}(s) = \tilde{F}(s)\int_{R^+}T^*(x)d\Phi_s(x).$$

Using the identity

$$\int_0^x se^{sy}\int_y^\infty e^{-st}\mu(t)dtdy = e^{sx}\int_x^\infty e^{-st}\mu(t)dt - \int_0^\infty e^{-st}\mu(t)dt$$

$$+ \int_0^x \mu(t)dt,$$

obtained by integration by parts, we can express ϕ_s in the form

$$\phi_s(x) = \phi_s(0) + e^{sx}\int_x^\infty e^{-st}\mu(t)dt - \int_0^\infty e^{-st}\mu(t)dt$$

$$= \phi_s(0) + \int_0^x se^{sy}\int_y^\infty e^{-st}\mu(t)dt - \int_0^x \mu(t)dt$$

$$= \phi_s(0) + \int_0^x s\phi_s(y)-\mu(y)dy.$$

It follows that

$$(20) \qquad s\hat{\mu}(s)\tilde{R}(s) = \tilde{F}(s)\int_{R^+} T^*(x)(s\phi_s(x)-\mu(x))dx$$

$$= \tilde{F}(s)\int_{R^+} T(x)(s\phi_s(x)-\mu(x))dx.$$

Let $s \downarrow 0$ in (20). Using monotone convergence on the left-hand side and bounded convergence on the right-hand side, we see from the Abelian theorem for Laplace transforms that

$$(21) \qquad \mu(\infty)\int_{R^+} R(t)dt = \int_{R^+} T(x)(\mu(\infty)-\mu(x))dx.$$

If $\mu(\infty) = 0$, then by the monotonicity of μ, $\mu(b) = 0$ for all b. As we have seen, this implies $p(b) = 0$ identically in b. Assume then that $\mu(\infty) > 0$. We shall show that under this assumption F and G degenerate at a common point.

Since $\bar{\eta} = \int_{R^+} T(x)dx < \infty$, it follows from (21) that

$$\mu(\infty) \left(\int_{R^+} T(x)dx - \int_{R^+} R(t)dt\right) = \int_{R^+} T(x)\mu(x)dx \geq 0.$$

But $\mu(\infty) > 0$, and by hypothesis

$$\bar{\eta} = \int_{R^+} T(x)dx \leq \int_{R^+} R(t)dt = \bar{\xi},$$

so that we must have

$$\bar{\eta} = \bar{\xi},$$

and

$$\int_{R^+} T(x)\mu(x)dx = \int_{R^+} (1-G(x))\mu(x)dx = 0.$$

Put

$$x_\mu = \inf \left\{x: \mu(x) > 0\right\}$$

and

$$x_G = \inf \left\{x: G(x) = 1\right\}.$$

Since

$$\mu(x) \uparrow \mu(\infty) > 0,$$

it follows that

(23) $$0 \leq x_\mu < \infty,$$

and

(24) $$\mu(x) \left\{ \begin{matrix} = \\ > \end{matrix} \right\} 0 \quad \text{according as } x \left\{ \begin{matrix} < \\ > \end{matrix} \right\} x_\mu .$$

Similarly, the monotonicity of G implies that

(25) $$G(x) \left\{ \begin{matrix} < \\ = \end{matrix} \right\} 1 \quad \text{according as } x \left\{ \begin{matrix} < \\ \geq \end{matrix} \right\} x_G .$$

Equation (25) and the relation

$$\int_{x_\mu}^{\infty}(1-G(x))\mu(x)dx = \int_{R^+}(1-G(x))\mu(x)dx = 0$$

show that

(26) $$0 \leq x_G \leq x_\mu .$$

If $x_G = 0$, then $G(x) = 1$ for $x \geq 0$. In this case $\overline{\eta} = \overline{\xi} = 0$, i.e., F and G degenerate at the origin. Assume, therefore, that $x_G > 0$. Put

(27) $$\omega(t) = \int_{R^+}\mu(t+x)dG(x) = \int_{I_{x_G}}\mu(t+x)dG(x).$$

It follows from (24) and (27) that $\omega(t) = 0$ if $t < x_\mu - x_G$. Let

$$t = x_\mu - x_G + h,$$

where $h > 0$ is arbitrary, and let h' be any number in the interval $0 < h' < \min(h,x_G)$. Then, from (24), (25), (27), and the monotonicity of μ we obtain

$$\omega(t) = \omega(x_\mu - x_G + h) = \int_{I_{x_G}}\mu(x_\mu - x_G + h + x)dG(x)$$

$$\geq \int_{I_{x_G - h', x_G}}\mu(x_\mu - x_G + h + x)dG(x) \geq \mu(x_\mu + h - h')(1-G(x-h')) > 0.$$

In summary,

(28) $$\omega(t) \begin{Bmatrix} = \\ > \end{Bmatrix} 0 \quad \text{according as } t \begin{Bmatrix} < \\ > \end{Bmatrix} x_\mu - x_G .$$

From (13) and Fubini's theorem we obtain

(29) $$\mu(b) = \int_{I_b} \omega(b-t)dF(t).$$

For all b in the interval

$$x_\mu - x_G \leq b < x_\mu,$$

(24), (28), and (29) imply

$$0 = \int_{I_b} \omega(b-t)dF(t) = \int_{I_{x_G-(x_\mu-b)}} \omega(b-t)dF(t) \geq \int_{I'_{x_G-(x_\mu-b)}} \omega(b-t)dF(t) \geq 0,$$

where

$$I'(x) = I_x - \{ x \} = [0,x).$$

Since $\omega(b-t)$ is positive for $t \in I'_{x_G-(x_\mu-b)}$, and $x_\mu-b$ can be made arbitrarily small by choosing b sufficiently close to x_μ, we must have

(30) $$F(x_G-0) = 0.$$

Moreover, (25) and (30) imply that

(31) $$x_G \geq \int_{I_{x_G}} xdG(x) = \bar{\eta} = \bar{\xi} = \int_{R^+-I'_{x_G}} xdF(x) \geq x_G,$$

so that

(32) $$\bar{\eta} = x_G = \bar{\xi}.$$

It is evident from (31) and (32) that

$$F(x_G) - F(x_G-0) = 1 = G(x_G) - G(x_G-0),$$

which completes the proof. It is evident from the physical meaning of the quantities that when the service time and spacing are almost surely equal to x_G, $p(b) = 1$ for $b \geq x_G$(1).

In the previous theorem the restriction that $\bar{\eta}$ is finite leaves open the question of how $p(b)$ behaves when $\bar{\xi} = \bar{\eta} = \infty$. This is resolved in theorem 7 whose proof hinges on the relation between the generating functions of the number of elements served in the processes in which $\alpha (0<\alpha\leq 1)$ and 1 are, respectively, the absorption probabilities.

We shall refer to the process in which α is the absorption probability and b the distance of the first element from 0 at time $t = 0$ as the α,b-process. Let $\pi_n^{(\alpha)}(b)$ denote the probability of serving exactly n elements in this process, i.e.,

$$\pi_n^{(\alpha)}(b) = p_n^{(\alpha)}(b) - p_{n+1}^{(\alpha)}(b) \quad , \quad n = 0,1,\dots \quad .$$

Let $\Pi^{(\alpha)}(x,b)$ be the generating function of $\left\{\pi_n^{(\alpha)}(b)\right\}_{n=0}^{\infty}$.

THEOREM 6(2). If $\alpha > 0$,

$$\Pi^{(\alpha)}(x,b) = \frac{\alpha \Pi^{(1)}(x,b)}{1-(1-\alpha)\int_{R^+} \Pi^{(1)}(x,t)dG(t)}$$

Proof: In the α,b-process let

$A_k = \left\{\text{elements } E_i, i=1,\dots,k, \text{ are serviced, } E_{k+1} \text{ is not serviced}\right\}$,
$B_{k+1} = \left\{\text{server is absorbed at arrival of } E_{k+1} \text{ at barrier}\right\}$,
$D_k = \left\{\text{exactly } k \text{ elements are serviced}\right\}$, $k \geq 0$,
and

(1) The analytic proof of this fact is easily constructed using the quantities p_k introduced in § 2.

(2) This result is due to C. Siegel.

$C_{k,n} = \left\{\text{exactly } n\text{-}k \text{ of the elements in the set } \{E_i\}_{i=k+2}^{\infty} \text{ are serviced}\right\}$, $\quad n \geq 0,\ k = 0,1,\ldots,n.$

Evidently D_n is the disjoint union

$$D_n = A_n B_{n+1} \cup \bigcup_{k=0}^{n} A_k \overline{B}_{k+1} C_{k,n} \quad ,$$

where $\overline{B}_{k+1}$ is the complement of B_{k+1}. Hence

$$P(D_n) = \pi_n^{(\alpha)}(b) = P(A_n)P(B_{n+1}|A_n)$$

$$+ \sum_{k=0}^{n} P(A_k)P(\overline{B}_{k+1}|A_k)P(C_{k,n}|A_k\overline{B}_{k+1}).$$

From (3) it follows that the probabilities $\{\pi_k^{(1)}(b)\}_{k=1}^{\infty}$ satisfy the functional equation

(33) $$\pi_k^{(1)}(b) = \int_{R^+}\int_{I_b} \pi_{k-1}^{(1)}(b-t+x)dF(t)dG(x),$$

with the initial condition

(34) $$\pi_0^{(1)}(b) = 1-F(b).$$

An argument analogous to the one leading to (1) shows that the sequence $\{P(A_k)\}_{k=0}^{\infty}$ also satisfies (33) and (34). Since these equations have a unique solution, we conclude that

$$P(A_k) = \pi_k^{(1)}(b).$$

Noting that

$$P(C_{k,n}|A_k\overline{B}_{k+1}) = \int_{R^+} \pi_{n-k}^{(\alpha)}(x)dG(x)$$

and

$$P(B_{n+1}|A_n) = \alpha,$$

we obtain

$$(35)\qquad \pi_n^{(\alpha)}(b) = \alpha\pi_n^{(1)}(b) + (1-\alpha)\sum_{k=0}^{n}\pi_k^{(1)}(b)\int_{R^+}\pi_{n-k}^{(\alpha)}(x)dG(x).$$

In terms of generating functions (35) becomes

$$(36)\qquad \Pi^{(\alpha)}(x,b) = \alpha\,\Pi^{(1)}(x,b) + (1-\alpha)\,\Pi^{(1)}(x,b)V^{(\alpha)}(x),$$

where

$$(37)\qquad V^{(\alpha)}(x) = \sum_{n=0}^{\infty}\int_{R^+}\pi_n^{(\alpha)}(b)dG(b)x^n.$$

But an interchange in the order of summation and integration in (37) shows that

$$(38)\qquad V^{(\alpha)}(x) = \int_{R^+}\Pi^{(\alpha)}(x,b)dG(b).$$

It follows from (36) and (38) that $V^{(\alpha)}(x)$ satisfies the equation

$$(39)\qquad V^{(\alpha)}(x) = \alpha V^{(1)}(x) + (1-\alpha)V^{(1)}(x)V^{(\alpha)}(x),$$

whose solution is

$$(40)\qquad V^{(\alpha)}(x) = \frac{\alpha V^{(1)}(x)}{1-(1-\alpha)V^{(1)}(x)} = \frac{\alpha\int_{R^+}\Pi^{(1)}(x,b)dG(b)}{1-(1-\alpha)\int_{R^+}\Pi^{(1)}(x,b)dG(b)}.$$

Substitution of this expression into (36) leads to the result asserted in the theorem.

COROLLARY 1. If, for some number b_0, $p^{(1)}(b_0) = 0$, then $p^{(\alpha)}(b_0) = 0$ for all $\alpha > 0$.

Proof: Since

$$\Pi^{(1)}(1,b_0) = 1 - p^{(1)}(b_0) = 1,$$

it follows from (40) that

$$\pi^{(\alpha)}(1,b_0) = 1 - p^{(\alpha)}(b_0) = 1.$$

COROLLARY 2. For each fixed b, $p^{(\alpha)}(b)$ is a continuous function of α in the interval $0 < \alpha \leq 1$.

Returning now to the problem of infinite expectations, we have the following result:

THEOREM 7. If $\alpha > 0$ and $\bar{\xi} = \infty$, then $p(b) = 0$ for all $b \geq 0$.

Proof: If $\bar{\eta} < \infty$, the assertion of the theorem is contained in theorem 5. Assume, therefore, that

$$\bar{\xi} = \bar{\eta} = \infty.$$

By corollary 1 of the preceding theorem, we assume without loss of generality that $\alpha = 1$.

Let

$$G_n(x) = \begin{cases} G(x) & , \quad x < n \\ 1 & , \quad x \geq n, \end{cases}$$

and denote by $p_{k,n}(b)$, $k \geq 0$, $n \geq 1$, the probability of serving at least k elements in the 1,b-process for which the spacing between consecutive members of the queue is governed by the distribution function G_n. Since G and G_n induce the same measure on the Borel subsets of $I_n' = [0,n)$, we obtain from (3) the inequality

$$(41) \qquad p_k(b)-p_{k,n}(b) \leq \int_{I_n'} dG(x) \int_{I_b} p_{k-1}(b-t+x)-p_{k-1,n}(b-t+x)dF(t)$$

$$+ F(b)(1-G(n-0)).$$

It follows inductively from (41) that for all k,

$$(42) \qquad p_k(b)-p_{k,n}(b) \leq \frac{F(b)}{1-F(b)} (1-G(n-0)).$$

For,

$$p_0(b) = p_{0,n}(b) = 1,$$

and taking (42) as inductive hypothesis,

$$p_{k+1}(b)-p_{k+1,n}(b) \leq \frac{F^2(b)}{1-F(b)} G(n-0)(1-G(n-0))$$

$$+ F(b)(1-G(n-0)) = \frac{F(b)}{1-F(b)} (1-G(n-0))[1-F(b)(1-G(n-0))]$$

$$\leq \frac{F(b)}{1-F(b)} (1-G(n-0)).$$

Since $\lim\limits_{n\to\infty} G(n-0) = 1$, and the hypothesis $\overline{\zeta} = \infty$ implies that $F(b) < 1$ for $b < \infty$, we conclude that

$$\lim_{n\to\infty} p_{k,n}(b) = p_k(b)$$

uniformly in k. It follows that

$$(43) \quad p(b) = \lim_{k\to\infty} p_k(b) = \lim_{k\to\infty} \lim_{n\to\infty} p_{k,n}(b) = \lim_{n\to\infty} \lim_{k\to\infty} p_{k,n}(b).$$

But $\lim\limits_{k\to\infty} p_{k,n}(b)$ is the probability of serving infinitely many elements in the process in which G_n is the governing distribution; hence by theorem (5),

$$\lim_{k\to\infty} p_{k,n}(b) = 0,$$

which completes the proof.

We have seen in theorems 5 and 7 that if $\bar{\xi} \geq \bar{\eta}$, then, barring an exceptional case, $p(b) \equiv 0$. We shall show in theorem 9 that when $\bar{\xi} < \bar{\eta}$, $p(b)$ can be made arbitrarily close to unity by choosing b sufficiently large. Thus the asymptotic behavior of $p(b)$ is sensitive to the relation between the first moments of the service time and spacing distributions. In contrast to this, as is proved below, if $F(\infty) = 1$, then for each fixed k, $p_k(b) \sim 1$ as $b \longrightarrow \infty$.

THEOREM 8. If $F(\infty) = 1$, then for $k \geq 0$,

$$\lim_{b\to\infty} p_k(b) = 1.$$

If $F(\infty) < 1$ and $\alpha > 0$, then for $k \geq 1$,

$$\lim_{b\to\infty} p_k(b) < 1.$$

Proof: From the functional equation satisfied by p_k it follows easily that

$$p_k(\infty) = p_{k-1}(\infty)F(\infty) + (1-\alpha)(1-F(\infty))\int_{R^+} p_k(x)dG(x), \tag{44}$$

where

$$p_k(\infty) = \lim_{b\to\infty} p_k(b).$$

If $F(\infty) = 1$, (44) reduces to

$$p_k(\infty) = p_{k-1}(\infty),$$

from which it follows inductively (since $p_0(b) \equiv 1$) $p_k(\infty) = 1$ for all k. Similarly, if $F(\infty) < 1$ and $\alpha > 0$, it is easy to calculate from (44) that $p_1(\infty) < 1$. Induction then shows

that $p_k(\infty) < 1$ for all $k \geq 1$.

THEOREM 9. If $\overline{\xi} < \overline{\eta}$, then

$$p(\infty) = \lim_{b \to \infty} p(b) = 1.$$

Proof: Since $p(b)$ is non-increasing in α, it suffices to prove the theorem for $\alpha = 1$. We note further that the hypothesis implies $\overline{\xi} < \infty$, and consequently $F(\infty) = 1$.

For $s > 0$ put

$$q_k(x) = 1 - p_k(x),$$

$$\phi_{s,k}(x) = \begin{cases} \int_{R^+} e^{-st} q_k(t+x)dt & , \quad x \geq 0 \\ \\ \phi_{s,k}(0) & , \quad x < 0, \end{cases}$$

and

$$\tilde{q}_k(s) = \int_{R^+} e^{-sx} q_k(x)dx.$$

Since $q_0(x) = 1-p_0(x) \equiv 0$, we have

(45) $$\phi_{s,0}(x) \equiv 0.$$

Replacing p_k by $1-q_k$ in (3), setting $\alpha = 1$, and taking Laplace transforms, we obtain by the same argument used to derive (15)

(46) $$\tilde{q}_k(s) = \tilde{F}(s)\int_{R^+} \phi_{s,k-1}(x)dG(x) + \tilde{R}(s) \quad , \quad k \geq 1.$$

From (45), (46), and the finiteness of $\overline{\xi}$ it follows inductively that

(47) $$\int_{R^+} q_k(x)dx < \infty \quad , \quad k \geq 0,$$

which is indispensible to the rest of the argument.

Defining

$$\phi^{-}_{s,k}(x) = -\phi_{s,k}(x),$$

it is obvious that

(i) $\phi^{-}_{s,k}(x) \leq \phi^{-}_{s,k}(x')$ if $x \leq x'$,

and

(ii) $\phi^{-}_{s,k}(x+0) = \phi^{-}_{s,k}(x)$.

Moreover, by theorem 8 and the Lebesgue bounded convergence theorem it follows that

(iii) $\lim\limits_{x\to\infty} \phi^{-}_{s,k}(x) = 0$.

Integrating (46) by parts (after replacing $\phi_{s,k-1}$ by $\phi^{-}_{s,k-1}$ and G by the left-continuous function G^{*}) we obtain

$$(48) \qquad \tilde{q}_k(s) = -\tilde{F}(s)\int_{R^+} G^{*}(x)\,d\phi^{-}_{s,k-1}(x) + \tilde{R}(s)$$

$$= -\tilde{F}(s)(-\tilde{q}_{k-1}(s) + \int_{R^+} T^{*}(x)\,d\phi^{-}_{s,k-1}(x)) + \tilde{R}(s),$$

where we recall that $T^{*} = 1-G^{*}$. By the same argument used to derive (20) from (19) it follows from (48) that

$$(49) \qquad \tilde{q}_k(s) = \tilde{F}(s)[\tilde{q}_{k-1}(s)+\int_0^\infty T(x)(s\phi_{s,k-1}(x)-q_{k-1}(x))dx]$$

$$+ \tilde{R}(s).$$

To evaluate the asymptotic behavior of (49) as $s\downarrow 0$, we note that

$$\int_{R^+} T(x)s\phi_{s,k-1}(x)dx \leq \int_{R^+} q_k(x)dx < \infty;$$

hence by bounded convergence and the Abelian theorem for Laplace transforms,

$$\lim_{s\downarrow 0} \int_{R^+} T(x)s\phi_{s,k-1}(x)dx = \lim_{x\to\infty} q_k(x) = 0.$$

Letting $s\downarrow 0$ in (49) and observing that q_k is non-decreasing in k, we find that

(50) $$\int_{R^+} q_k(x)dx - \int_{R^+} q_{k-1}(x)dx = \overline{\xi} - \int_{R^+} T(x)q_{k-1}(x)dx \geq 0.$$

Now letting $k \longrightarrow \infty$, it follows by monotone convergence that

(51) $$\int_{R^+} T(x)(1-p(x))dx \leq \overline{\xi}.$$

If $\overline{\eta} = \infty$, the theorem follows at once from (51); for, by the monotonicity of $p(x)$ we have

$$\overline{\eta}(1-p(\infty)) \leq \overline{\xi},$$

which contradicts the hypothesis unless $p(\infty) = 1$.

Suppose now that $\overline{\eta} < \infty$. Since we are dealing with the case $\alpha = 1$, the function $\mu(b)$ introduced in the proof of theorem 4 coincides with $p(b)$. We may therefore replace μ by p in (21) (which was derived under the hypothesis $\overline{\eta} < \infty$), obtaining the relation

$$\int_{R^+} p(x)T(x)dx = p(\infty)(\overline{\eta}-\overline{\xi}).$$

On the other hand, from (51)

$$\int_{R^+} p(x)T(x)dx \geq \overline{\eta} - \overline{\xi}.$$

The last two equations are compatible only if $p(\infty) = 1$.

It may be of some interest to note that this proof contains a lower bound for $p(b)$:

COROLLARY. For all distribution functions F and G,

$$p(b) \geq 1 - \frac{\bar{\xi}}{\int_{I_b} T(x)dx} .$$

Proof: If $\bar{\xi} = \infty$, the statement is trivially true. If $\bar{\xi} < \infty$, it is an immediate consequence of (51), whose derivation depends on the finiteness of $\bar{\xi}$ but not on the relation between $\bar{\xi}$ and $\bar{\eta}$. Thus, (51) and the monotonicity of p imply that

$$(1-p(b))\int_{I_b} T(x)dx \leq \int_{I_b} T(x)(1-p(x))dx \leq \bar{\xi},$$

which completes the proof.

The previous theorems describe the behavior of $p(b)$ when $\alpha > 0$. We conclude with two results which round out the treatment by covering the exceptional cases.

THEOREM 10. If $\alpha = 0$ and $\int_{R^+} F(x)dG(x) > 0$, then $p(b) = 1$ for all b.

Proof: This follows inductively from (3) and (4).

THEOREM 11. If $\int_{R^+} F(x)dG(x) = 0$, then $p(b) = 0$ for all b.

Proof: We showed earlier (cf. p. 8,9) that the hypothesis implies that

$$\int_{R^+} p_k(x)dG(x) = 0 \quad , \quad k \geq 1.$$

By bounded convergence,

$$\int_{R^+} p(x)\,dG(x) = 0,$$

which means that p(b) satisfies the equation

$$p(b) = \int_{R^+}\int_{I_b} p(b-t+x)\,dF(t)\,dG(x).$$

Moreover, under the hypothesis it is clear that $\bar{\xi} \geq \bar{\eta}$, and there cannot exist a point x_o at which both F and G degenerate. When these conditions hold, as is shown in the proof of theorem 5, the above equation has the unique solution p(b) = 0 for all b.

BIBLIOGRAPHY

[1] Borel, E., Sur l'emploi du théorème de Bernoulli pour faciliter le calcul d'une infinité de coefficients. Application au problème de l'attente à un guichet, C. R. Acad. Sci. Paris, Vol. 214, 1942, pp. 452-456.

[2] Hirsch, W. M., Conn, J., and Siegel, C. A queueing process with an absorbing state, Comm. on Pure and Applied Mathematics, Vol. XIV, 1961, pp. 137-153.

[3] Karlen, S., Miller, Jr., R. G., and Prabhu, N. V., Note on a moving single server problem, Ann. Math. Statistics, Vol. 30, 1959, pp. 243-246.

[4] McMillan, B., and Riordan, J., A moving single server problem, Ann. Math. Statistics, Vol. 28, 1957, pp. 471-478.

[5] Saks, S., Théorie de l'Intégrale, Warsaw, 1937.

[6] Takacs, L., Investigation of waiting time problems by reduction to Markov processes, Acta Math. Acad. Sci. Hungaricae, Vol. 6, 1955, pp. 101-129.

[7] Takacs, L., Introduction to the Theory of Queues, New York, 1962.

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